Integrating Factor

For a differential equation:

$$\frac{dy}{dx} + P(x)\,y(x) = Q(x),$$

Multiply the equation by $\mu(x)$ ← Integrating factor

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y(x) = \mu(x)\,Q(x)$$

If "$\text{LHS} = \frac{d}{dx}\left(\mu(x)\,y(x)\right)$",

Then we have:

$$\frac{d}{dx}\left(\mu(x)\,y(x)\right) = \mu(x)\,Q(x)$$

$$\Rightarrow \quad \mu(x)\,y(x) = \int \mu(x)\,Q(x)\,dx + C$$

$$\Rightarrow \quad y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,Q(x)\,dx + C\right]$$

The condition " LHS $= \frac{d}{dx}(\mu(x)y(x))$ ":

$$\mu(x)\, y'(x) + \mu(x)\, P(x)\, y(x) = \mu(x)\, y'(x) + \mu'(x)\, y(x)$$

$$\therefore \quad \mu(x)\, P(x) = \mu'(x)$$

$$P(x) = \frac{\mu'(x)}{\mu(x)}$$

$$\int P(x)\, dx = \int \frac{\mu'(x)}{\mu(x)}\, dx$$

$$\int P(x)\, dx = \ln \mu(x)$$

$$\mu(x) = e^{\int P(x)\, dx}$$

Example :

$$\begin{cases} (x+1)\dfrac{dy}{dx} + 2y = x, & x > -1 \\ y(0) = 0 \end{cases}$$

$$\frac{dy}{dx} + \frac{2}{x+1} y = \frac{x}{x+1} \quad — \quad (1)$$

Let $\mu(x) = e^{\int \frac{2}{x+1} dx}$

$$= e^{2\ln(x+1)} = (x+1)^2$$

$(1) \times \mu(x)$,

$$(x+1)^2 \frac{dy}{dx} + 2(x+1)y = (x+1)x$$

$$\frac{d}{dx}(\mu(x) y(x)) = x^2 + x$$

$$\mu(x) y(x) = \int x^2 + x \, dx + C$$

$$y(x) = \frac{1}{(x+1)^2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)$$

$$y(0) = 0$$

$\Rightarrow C = 0$.

$$y(x) = \frac{1}{(x+1)^2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$$

2^{nd} Order Integrating Factor

In the Form:

$$\frac{d^2y}{dx^2} - Cy = 0 \quad (\text{homogeneous})$$

or $$\frac{d^2y}{dx^2} - Cy = f(x) \quad (\text{inhomogeneous})$$

with conditions: $y(x_0) = a$, $y'(x_1) = b$

For homogenous Case:

$$\frac{d^2y}{dx^2} = Cy$$

Multiply both sides by $\frac{dy}{dx}$ ← Integrating Factor.

$$\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} = c\left(y \cdot \frac{dy}{dx}\right) \quad - (2)$$

Note $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2}$$

and $$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

(2): $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = c\frac{d}{dx}(y^2)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = c\frac{d}{dx}(y^2)$$

$$\int \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 dx = c\int \frac{d}{dx}(y^2)\, dx$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = c\,y^2 + D.$$

$$\frac{dy}{dx} = \pm\sqrt{c\,y^2 + D}$$

Let's assume $D = 0$ for simplicity.

If we can solve the ODE
under the assumption,
by Uniqueness of ODE solution,
we can conclude that is the solution.

$D = 0$,

$$\frac{dy}{dx} = \pm\sqrt{c}\,y$$

$$\int \frac{dy}{y} = \int \sqrt{c}\, dx$$

$$\ln y = \sqrt{c}\,x + E$$

$$y = e^{\sqrt{c}\, x + \bar{c}}$$

$$y(x) = A_1 e^{\sqrt{c}\, x} \quad \text{or} \quad A_2 e^{-\sqrt{c}\, x}$$

Note If y_1, y_2 satisfy

$$\frac{d^2 y}{dx^2} - C y = 0,$$

$$\frac{d^2 (y_1 + y_2)}{dx^2} - C (y_1 + y_2) = 0$$

$$\therefore \quad y(x) = A_1 e^{\sqrt{c}\, x} + A_2 e^{-\sqrt{c}\, x}$$

is a general solution

$$y(x_0) = a, \quad y'(x_1) = b$$

$\Rightarrow$ solve for A_1, A_2.

For inhomogeneous Case :

$$\begin{cases} \dfrac{d^2y}{dx^2} - cy = f(x) , \\ y(x_0) = a , \quad y'(x_1) = b . \end{cases}$$

Split the problem into two parts :

$$\frac{d^2y_1}{dx^2} - cy_1 = 0$$

and $$\frac{d^2y_2}{dx^2} - cy_2 = f(x)$$

Note $y = y_1 + y_2$ is also a solution

we solve y_1 by homogeneous case.

For y_2,

We "guess" the solution.

For example,

$$\frac{d^2 y_2}{dx^2} - y_2 = x^2$$

We guess $y_2 = \alpha x^2 + \beta x + \gamma$

$$\frac{d^2 y_2}{dx^2} = 2\alpha$$

$$\frac{d^2 y_2}{dx^2} - y_2 = -\alpha x^2 - \beta x + (2\alpha - \gamma)$$

Comparing coefficient:

$-\alpha = 1$, $-\beta = 0$, $2\alpha - \gamma = 0$

$$\Rightarrow \begin{cases} \alpha = -1, \\ \beta = 0, \\ \gamma = -2 \end{cases}$$

$\therefore$ $y_2(x) = -x^2 - 2$

$y_1(x) = \alpha_1 e^{x} + \alpha_2 e^{-x}$

$$y(x) = \alpha_1 e^{x} + \alpha_2 e^{-x} - x^2 - 2$$

Solve α_1, α_2

by $y(x_0) = a$, $y'(x_1) = b$.

Exercise:

(a) $y' + 2xy = x$

(b) $y'' - 4y = e^{-3x}$

Exercise Solutions

(c). $y' + 2xy = x$

$$\mu(x) := e^{\int 2x\,dx} = e^{x^2}$$

$$\mu(x) y' + 2x\mu(x) y = x e^{x^2}$$

$$(\mu(x) y(x))' = x e^{x^2}$$

$$\mu(x) y(x) = \int x e^{x^2} dx$$

$$\mu(x) y(x) = \frac{1}{2}\int e^{x^2} d(x^2)$$

$$\mu(x) y(x) = \frac{1}{2} e^{x^2} + C$$

$$y(x) = e^{-x^2}\left(\frac{1}{2} e^{x^2} + C\right)$$

$$= \frac{1}{2} + Ce^{-x^2}$$

(b).

$$y'' - 4y = e^{-3x}$$

homogeneous solution :

$$\frac{d^2 y}{dx^2} = 4y$$

$$\Rightarrow \frac{dy}{dx} \frac{d^2 y}{dx^2} = 4y \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 = 4 \frac{d}{dx} (y^2)$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = 4y^2 + C$$

Take $C = 0$,

$$\frac{dy}{dx} = \pm 2y$$

$$y = \alpha_1 e^{2x} + \alpha_2 e^{-2x}$$

Inhomogeneous Part:

Guess $y(x) = \beta e^{-3x}$

$$y'' - 4y = 9\beta e^{-3x} - 4\beta e^{-3x}$$

Comparing coef.:

$$1 = 5\beta \Rightarrow \beta = \frac{1}{5}.$$

$\therefore$ A solution is

$$y(x) = \alpha_1 e^{2x} + \alpha_2 e^{-2x} + \frac{1}{5} e^{-3x}$$

α_1, α_2 solved by $y(x_0) = a$, $y'(x_1) = b$.